Thermodynamics of a Bose-Einstein Condensate with Weak Disorder

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We consider the thermodynamics of a homogeneous superfluid dilute Bose gas in the presence of weak quenched disorder. Following the zero-temperature approach of Huang and Meng, we diagonalize the Hamiltonian of a dilute Bose gas in an external random delta-correlated potential by means of a Bogoliubov transformation. We extend this approach to finite temperature by combining the Popov and the many-body T−matrix approximations. This approach permits us to include the quasi-particle interactions within this temperature range. We derive the disorder-induced shifts of the Bose-Einstein critical temperature and of the temperature for the onset of superfluidity by approaching the transition points from below, i.e., from the superfluid phase. Our results lead to a phase diagram consistent with that of the finite-temperature theory of Lopatin and Vinokur which was based on the replica method, and in which the transition points were approached from above.

I. INTRODUCTION

An interacting ultracold dilute Bose gas in a weak random external potential, which is homogeneous in the mean, represents an interesting model for studying the relation between Bose-Einstein condensation and superfluidity and has been the subject of various theoretical investigations in the last few years [1, 2, 3, 4, 5, 6]. Two different methods have been used for performing the average over the impurity scatterers. In Refs. [1, 2, 3] the average of the grand potential over the disorder is taken perturbatively in the strength of the disorder after diagonalizing the Hamiltonian by means of a Bogoliubov transformation. Alternatively, in Refs. [4, 5, 6] the averaging is implemented by the replica method. In Ref. [4], the replica symmetric solution of the model is found by a systematic diagrammatic Beliaev-Popov perturbation theory for the dilute superfluid gas in the presence of disorder. At zero temperature the two different approaches give equivalent results. At finite temperatures, however, the approach based on the Bogoliubov transformation becomes unsatisfactory since it necessarily neglects important quasi-particles correlations [1, 3]. The replica trick, on the other hand, has limitations of it's own by making the mathematical and also the physical description less transparent. For these reasons it would clearly be desirable to develop an alternative theory avoiding these shortcomings. This is the goal of this paper, where we show how the perturbative approach of Refs. [1, 2, 3] can be extended to include the leading effects of the scattering between quasi-particles at finite temperature. As a result, some discrepancies between the two different methods are resolved.

The paper is organized as follows. In order to make the paper self-contained, we briefly rederive in Section II the thermodynamic potential and the equation of state for the case of a vanishing spatial correlation length of the disorder potential [1]. In the limit of zero temperature, we also determine the high-order Beliaev corrections to the chemical potential [4]. In Section III we give the derivation of the superfluid component of the system in the presence of a random potential. The disorder-induced corrections of the velocity of sound are there obtained from hydrodynamic equations for the superfluid component of the system. In Section IV we extend the theory to finite temperatures within the mean-field Popov approximation. In Section V the thermodynamics and the phase diagram are investigated by means of the many-body T−matrix approximation. In particular, we use the theory to calculate the shift of the Bose-Einstein critical temperature and of the temperature for the loss of the superfluidity when approaching the critical points from below. Comments and conclusions remain for Section VI.

II. BOGOLIUBOV'S THEORY

We consider the effects of an external random field on the thermodynamics of a dilute Bose gas. The random field is assumed to have a probability distribution $P[U]$ normalized to one when averaged over all disorder configurations, that is $\int d\,[U]\,P[U] = 1$. The average over the disorder fields is defined as

$$\langle \bullet \rangle = \int d\,[U] \bullet P[U], \tag{1}$$

and for the disorder potential we assume the ensemble averages

$$\begin{aligned} \langle U(\mathbf{x}) \rangle &= 0, \\ \langle U(\mathbf{x}) U(\mathbf{x}') \rangle &= R\,(\mathbf{x} - \mathbf{x}')\,. \end{aligned} \tag{2}$$

We assume that any time scale of the disorder potential is frozen, i.e. very long in comparison with the thermodynamic time scale. This so-called quenched-disorder limit has the consequence that the disorder average must be taken after the thermodynamic average over the grand-canonical ensemble. Therefore, the total average of an observable reads

$$\langle \langle O \rangle_{\mathrm{gr}} \rangle = \int d\,[U]\,P[U] \langle O(\Psi) \rangle_{\mathrm{gr}}, \tag{3}$$

where $\langle O(\Psi)\rangle_{\rm gr}$ indicates the grand-canonical average. At very low temperatures, when the wavelength of the atoms in the gas is much larger than the range of the impurity scatterers responsible for the random potential [9], one may consider the limit of a $\delta-$correlated type of disorder $R(\mathbf{x}-\mathbf{x}') = R_0\ \delta(\mathbf{x}-\mathbf{x}')$. The constant R_0 is then related to the concentration of the impurities and to the $s-$wave scattering length of the random scatterers [9].

At equilibrium, the grand-canonical partition function of a Bose gas in a disordered medium is given as a functional of the random external potential $U(\mathbf{x})$

$$Z_{\rm gr}[U] = \int d[\psi^*]d[\psi] \exp\left\{-\frac{1}{\hbar}S[\psi^*,\psi;U]\right\}\ , \quad (4)$$

where the functional integral is performed over c-number fields $\psi^*(\mathbf{x},\tau)$ and $\psi(\mathbf{x},\tau)$ periodic in imaginary time over $\hbar\beta = \hbar/k_BT$. The Euclidean action S is given by

$$\begin{aligned} S[\psi^*,\psi;U] = & \quad (5)\\ \int_0^{\hbar\beta} d\tau \int d\mathbf{x}\ \psi^*(\mathbf{x},\tau) & \left\{\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu + U(\mathbf{x})\right]\right.\\ & \left. + \frac{1}{2}\int d\mathbf{x}'\ \psi^*(\mathbf{x}',\tau)V(\mathbf{x}-\mathbf{x}')\psi(\mathbf{x}',\tau)\right\}\psi(\mathbf{x},\tau)\ , \end{aligned}$$

where μ is the chemical potential and $V(\mathbf{x}-\mathbf{x}')$ the atomic repulsive interaction potential. The disorder average of the thermodynamic potential $\Omega = -(\ln Z_{\rm gr})/\beta$ is obtained from

$$\langle\Omega\rangle = -\frac{1}{\beta}\langle \ln Z_{\rm gr}\rangle. \quad (6)$$

The average of Eq. (6) is highly non-trivial because the disorder average is nonlinear in $Z_{\rm gr}$ due to the logarithm. In this paper we calculate this average by following the method of Huang and Meng in Ref. [1] which is based on a canonical Bogoliubov transformation [8].

Expanding the fields in Fourier modes in a finite volume V according to $\psi(\mathbf{x},\tau) = (1/\hbar\beta V)^{1/2}\sum_{\mathbf{k},n} a_{\mathbf{k},n}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_n\tau)}$ with the Matsubara frequencies $\omega_n = 2\pi n/\hbar\beta$, and the corresponding complex conjugate expression for $\psi^*(\mathbf{x},\tau)$, we write the action in momentum space as

$$\begin{aligned} S[a^*,a] = & \sum_{\mathbf{k},n}(-i\hbar\omega_n + \epsilon_\mathbf{k} - \mu)a^*_{\mathbf{k},n}a_{\mathbf{k},n} \quad (7)\\ & + \frac{1}{V}\sum_{\mathbf{k},\mathbf{k}',n} a^*_{\mathbf{k},n}U_{\mathbf{k}-\mathbf{k}'}a_{\mathbf{k}',n}\delta_{n,0}\\ & + \frac{1}{2\hbar\beta V}\sum_{\substack{\mathbf{k},\mathbf{k}',\mathbf{q}\\ n,n',m}} V_\mathbf{q}\ a^*_{\mathbf{k}+\mathbf{q},n+m}a^*_{\mathbf{k}'-\mathbf{q},n'-m}a_{\mathbf{k}',n'}a_{\mathbf{k},n}\ . \end{aligned}$$

In this equation we have introduced the free particle dispersion $\epsilon_\mathbf{k} = \hbar^2\mathbf{k}^2/2m$ and the Fourier transform of the interaction potential $V_\mathbf{q} = \int d\mathbf{x}\ V(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}}$. At very low temperatures, the de Broglie wavelength $\lambda_{\rm th}$ of the atoms is much larger than the range r_0 of the interaction potential such that $r_0/\lambda_{\rm th} \ll 1$. Therefore, only s-wave scattering is relevant in the gas and we can neglect the momentum dependence of the interaction potential, setting $V_\mathbf{q} = V_0$. Breaking the gauge-symmetry of the action by introducing the decomposition $\psi(\mathbf{x},\tau) = \sqrt{n_0} + \psi'(\mathbf{x},\tau)$ and expanding the resulting expression up to quadratic order in ψ' and U, the effective action becomes

$$\begin{aligned} S^{(2)}[a,a^*] = & -\hbar\beta\mu n_0 V + \frac{1}{2}\hbar\beta n_0^2 V_0 V + \hbar\beta n_0 U_\mathbf{0} \quad (8)\\ & + \sum_{\mathbf{k},n}{}^{'}(-i\hbar\omega_n + \epsilon_\mathbf{k} - \mu + 2n_0V_0)a^*_{\mathbf{k},n}a_{\mathbf{k},n}\\ & + \left(\frac{n_0}{V}\right)^{\frac{1}{2}}\sum_{\mathbf{k},n}{}^{'}\left(a^*_{\mathbf{k},n}U_\mathbf{k} + U_{-\mathbf{k}}a_{\mathbf{k},n}\right)\delta_{n,0}\\ & + \frac{1}{2}n_0V_0\sum_{\mathbf{k},n}{}^{'}\left(a^*_{\mathbf{k},n}a^*_{-\mathbf{k},-n} + a_{\mathbf{k},n}a_{-\mathbf{k},-n}\right)\ , \end{aligned}$$

where the prime denotes that $\mathbf{k} = \mathbf{0}$ is excluded from the sum. The condensate density n_0 remains to be determined by minimizing the thermodynamic potential. Note that we have also neglected as of higher order the term $1/V\sum'_{\mathbf{k},\mathbf{k}',n}a^*_{\mathbf{k},n}\ U_{\mathbf{k}-\mathbf{k}'}\ a_{\mathbf{k}',n}\delta_{n,0}$, which requires the assumption of weak disorder [1]. The effective action is diagonalized by a Bogoliubov transformation

$$\begin{aligned} a_{\mathbf{k},n} &= u_\mathbf{k}\alpha_{\mathbf{k},n} - v_\mathbf{k}\alpha^*_{-\mathbf{k},-n} - z_\mathbf{k}\\ a^*_{\mathbf{k},n} &= u^*_\mathbf{k}\alpha^*_{\mathbf{k},n} - v^*_\mathbf{k}\alpha_{-\mathbf{k},-n} - z^*_{-\mathbf{k}}, \end{aligned} \quad (9)$$

where the coherence factors $u_\mathbf{k}$, $v_\mathbf{k}$ and the complex number $z_\mathbf{k}$ can be taken real positive by appropriately choosing the phase of the complex fields. In such a case we have

$$\begin{aligned} u^2_\mathbf{k} &= \frac{1}{2}\left[1 + \frac{\epsilon_\mathbf{k} - \mu + 2n_0V_0}{\hbar\Omega_\mathbf{k}}\right]\\ v^2_\mathbf{k} &= \frac{1}{2}\left[-1 + \frac{\epsilon_\mathbf{k} - \mu + 2n_0V_0}{\hbar\Omega_\mathbf{k}}\right]\\ z_{\pm\mathbf{k}} &= (n_0/V)^{\frac{1}{2}}\frac{U_\mathbf{k}}{\hbar\Omega_\mathbf{k}}\left(u_\mathbf{k} - v_\mathbf{k}\right)^2, \end{aligned} \quad (10)$$

where $|u_\mathbf{k}|^2 - |v_\mathbf{k}|^2 = 1$, $u_\mathbf{k}v_\mathbf{k} \geq 0$, and where the Bogoliubov spectrum is given by

$$\hbar\Omega_\mathbf{k} = \sqrt{\left(\epsilon_\mathbf{k} - \mu + 2n_0V_0\right)^2 - \left(n_0V_0\right)^2}. \quad (11)$$

After the diagonalization the action reads

$$\begin{aligned} S^{(2)}[a,a^*] = & -\hbar\beta\mu n_0 V + \frac{1}{2}\hbar\beta n_0^2V_0V + \hbar\beta n_0U_\mathbf{0}\\ & + \sum_{\mathbf{k},n}{}^{'}\alpha^*_{\mathbf{k},n}\alpha_{\mathbf{k},n}\ \left(-i\hbar\omega_n + \hbar\Omega_\mathbf{k}\right)\\ & + \frac{\beta\hbar}{2}\sum_{\mathbf{k}}{}^{'}\left[\Omega_\mathbf{k} - (\epsilon_\mathbf{k} - \mu + 2n_0V_0)\right]\\ & - \frac{\beta\hbar}{V}\sum_{\mathbf{k},n}{}^{'}|U_\mathbf{k}|^2\frac{n_0\left(\epsilon_\mathbf{k} - \mu + n_0V_0\right)}{\hbar\Omega^2_\mathbf{k}}. \end{aligned} \quad (12)$$

By performing the functional integral of Eq. (4) and averaging over the disorder, the thermodynamic potential of Eq. (6) becomes

$$\begin{aligned}\langle\Omega\rangle =& -\mu n_0 V + \frac{1}{2}n_0^2 V_0 V + {\sum_{\mathbf{k}}}' \frac{1}{\beta}\ln\left(1-e^{-\beta\hbar\Omega_{\mathbf{k}}}\right) \\ &+ \frac{1}{2}{\sum_{\mathbf{k}}}' \left[\hbar\Omega_{\mathbf{k}} - (\epsilon_{\mathbf{k}} - \mu + 2n_0V_0)\right] \\ &- {\sum_{\mathbf{k}}}' n_0 R_0 \frac{(\epsilon_{\mathbf{k}} - \mu + n_0V_0)}{\hbar\Omega_{\mathbf{k}}^2}. \end{aligned} \tag{13}$$

The first two terms on the r.h.s. represent the mean-field result while the remaining three terms describe thermal and quantum fluctuations. From the thermodynamic relation $n = -(1/V)\,\partial\langle\Omega\rangle/\partial\mu$ we have for the particle-density in the grand-canonical ensemble

$$n = n_0 + n' + n_{\mathrm{R}}, \tag{14}$$

with the depletion due to the normal interaction

$$\begin{aligned} n' =& \frac{1}{V}{\sum_{\mathbf{k}}}' \left[\frac{\epsilon_{\mathbf{k}} - \mu + 2n_0V_0}{\hbar\Omega_{\mathbf{k}}} N(\hbar\Omega_{\mathbf{k}}) \right.\\ &\left. + \frac{\epsilon_{\mathbf{k}} - \mu + 2n_0V_0 - \hbar\Omega_{\mathbf{k}}}{2\hbar\Omega_{\mathbf{k}}}\right], \end{aligned} \tag{15}$$

and the disorder-induced depletion

$$n_{\mathrm{R}} = \frac{1}{V}{\sum_{\mathbf{k}}}' n_0 R_0 \frac{(\epsilon_{\mathbf{k}} - \mu + n_0V_0)^2}{\hbar\Omega_{\mathbf{k}}^4}. \tag{16}$$

Here $N(\hbar\Omega_{\mathbf{k}}) = (e^{\beta\hbar\Omega_{\mathbf{k}}} - 1)^{-1}$ is the Bose distribution function. The chemical potential can be eliminated from this expression by minimizing $\langle\Omega\rangle$, given in Eq. (13), with respect to the condensate density n_0 [cf. Eq. (21) below]. In the mean-field approximation this yields $\mu \simeq n_0V_0$, which corresponds to the Hugenholtz-Pines relation for that approximation. At this minimum, the thermodynamical potential of Eq. (13) becomes (with $n_0 \simeq \mu/V_0$)

$$\begin{aligned}\langle\Omega\rangle \simeq& -\frac{1}{2}n_0^2V_0V + {\sum_{\mathbf{k}}}' \frac{1}{\beta}\ln\left(1-e^{-\beta\hbar\Omega_{\mathbf{k}}}\right) \\ &+ \frac{1}{2}{\sum_{\mathbf{k}}}' \left[\hbar\Omega_{\mathbf{k}} - \epsilon_{\mathbf{k}} - n_0V_0\right] - {\sum_{\mathbf{k}}}' \frac{R_0 n_0}{\epsilon_{\mathbf{k}} + 2n_0V_0}, \end{aligned} \tag{17}$$

with $\hbar\Omega_{\mathbf{k}} \simeq \sqrt{\epsilon_{\mathbf{k}}\left(\epsilon_{\mathbf{k}} + 2n_0V_0\right)}$. Analogously, in the equation of state of Eq. (14), we get

$$n' \simeq n_0 \frac{8}{3}\left(\frac{n_0a^3}{\pi}\right)^{\frac{1}{2}} + \frac{1}{V}{\sum_{\mathbf{k}}}' \frac{\epsilon_{\mathbf{k}} + n_0V_0}{\hbar\Omega_{\mathbf{k}}}N(\hbar\Omega_{\mathbf{k}}), \tag{18}$$

and

$$n_{\mathrm{R}} \simeq \frac{1}{V}{\sum_{\mathbf{k}}}' \frac{R_0 n_0}{\left(\epsilon_{\mathbf{k}} + 2n_0V_0\right)^2}. \tag{19}$$

Performing analytically the integration in Eq. (19), the number of condensate particles depleted by the disorder becomes [1]

$$n_{\mathrm{R}} = R_0 \frac{m^2}{8\ \pi^{3/2}\ \hbar^4}\sqrt{\frac{n_0}{a}}, \tag{20}$$

where we have made the replacement $V_0 \to \mathrm{T}^{2B}$, that is, we have eliminated the unknown bare potential V_0 for the two-body scattering matrix $\mathrm{T}^{2B} \equiv 4\pi\hbar^2 a/m$ proportional to the $s-$wave scattering length a. According to the limitations of the Bogoliubov approximation, the theory is valid under the conditions $n_{\mathrm{R}}, n' \ll n_0 \simeq n$. The constraint $n' \ll n$ implies the diluteness condition $n^{1/3}a \ll 1$. The condition $n_{\mathrm{R}} \ll n$ is equivalent to the inequality $R_0' \equiv m^2 R_0/8\pi^{3/2}\hbar^4\,(na)^{1/2} \ll 1$. Associating to the strength R_0 of the disorder perturbation a length scale defined as $d \equiv \left(2\pi\hbar^2/m\right)^2/R_0$, the condition on R_0' can be rewritten as $2\left(n^{1/3}a\right)^{1/2} dn^{1/3}/\sqrt{\pi} \gg 1$. In the dilute limit $n^{1/3}a \ll 1$ this requires $dn^{1/3} \gg 1$, *i.e.* the length scale associated with the interaction energy due to the impurity potential must be much larger than the interparticle distance $n^{-1/3}$. The condition $R_0' \ll 1$ can also be reexpressed and elucidated by introducing the healing length of the condensate wave function. The latter is defined as $\xi_{\mathrm{heal}} \equiv 1/\sqrt{8\pi na}$. According to the theory of superfluidity in BEC, the inverse of the healing length characterizes the upper boundary of the momenta in the phononic spectrum of the fluid. At this wavelength, the energy of the excitations is of the order of $\hbar\Omega_{\mathbf{k}} \simeq \mu$. The condition $R_0' \ll 1$ is equivalent to $\sqrt{2}\pi\xi_{\mathrm{heal}} \ll d$. Therefore, the theory is valid when the energy of the excitations, induced by the impurity scattering, is far below the value $\hbar/\xi_{\mathrm{heal}}^2$ that marks the crossover from the collective phononic excitations to the single-particle excitations.

The lowest-order Hugenholtz-Pines condition $\mu = n_0V_0$, neglects the effects of quasi-particle interactions [10, 11] as well as the scattering between the quasi-particles and the impurities [4]. Nevertheless, the beyond mean-field Beliaev corrections to the leading order result $\mu = n_0V_0$ depend, both in the normal and in the disorder interactions, only on two-body collisions and can be calculated in the framework of the Bogoliubov theory [12]. Minimizing the thermodynamic potential of Eq. (13) with respect to the condensate density, we have

$$\begin{aligned}\mu =& n_0V_0 + V_0\frac{1}{V}{\sum_{\mathbf{k}}}' \left[\frac{2\epsilon_{\mathbf{k}} - 2\mu + 3n_0V_0}{\hbar\Omega_{\mathbf{k}}}N(\hbar\Omega_{\mathbf{k}}) \right.\\ &\left. + \frac{2\epsilon_{\mathbf{k}} - 2\mu + 3n_0V_0 - 2\hbar\Omega_{\mathbf{k}}}{2\hbar\Omega_{\mathbf{k}}}\right] \\ &- \frac{1}{V}{\sum_{\mathbf{k}}}' R_0 \left[\frac{\epsilon_{\mathbf{k}} - \mu + 2n_0V_0}{\hbar\Omega_{\mathbf{k}}^2} \right.\\ &\left. - \frac{2n_0V_0\left(\epsilon_{\mathbf{k}} - \mu + n_0V_0\right)\left(2\epsilon_{\mathbf{k}} - 2\mu + 3n_0V_0\right)}{\hbar\Omega_{\mathbf{k}}^4}\right]. \end{aligned} \tag{21}$$

In order to get the corrections to the mean-field result we substitute in the right-hand-side of equation Eq. (21) the zero-loop result $\mu = n_0 V_0$ to obtain the next order correction to the relation between the chemical potential and the condensate density. At $T=0$, the Bose distribution $N(\hbar\Omega_{\mathbf{k}})$ can be neglected, and we have

$$\mu = n_0 V_0 + 2V_0 \frac{1}{V}\sum_{\mathbf{k}}{}' \frac{\epsilon_{\mathbf{k}} + n_0 V_0 - \hbar\Omega_{\mathbf{k}}}{2\hbar\Omega_{\mathbf{k}}} - n_0 V_0^2 \frac{1}{V}\sum_{\mathbf{k}}{}' \frac{1}{2\hbar\Omega_{\mathbf{k}}} - \frac{1}{V}\sum_{\mathbf{k}}{}' R_0 \frac{\epsilon_{\mathbf{k}} - 2n_0 V_0}{(\epsilon_{\mathbf{k}} + 2n_0 V_0)^2}. \tag{22}$$

Subtracting the ultraviolet-divergent contribution $-n_0 V_0^2 (1/V)\sum_{\mathbf{k}}{}' 1/2\epsilon_{\mathbf{k}}$ in the third term and eliminating in the above expression the bare potential V_0 for the two-body scattering matrix defined by the Lippmann-Schwinger equation $\mathrm{T}^{2B} = V_0 - V_0(1/V)\sum_{\mathbf{k}}{}' \frac{1}{2\epsilon_{\mathbf{k}}}\mathrm{T}^{2B}$, we find

$$\mu = n_0 \mathrm{T}^{2B}\left(1 + \frac{40}{3}\sqrt{\frac{n_0 a^3}{\pi}}\right) - \frac{1}{V}\sum_{\mathbf{k}}{}' R_0 \frac{\epsilon_{\mathbf{k}}}{(\epsilon_{\mathbf{k}} + 2n_0 \mathrm{T}^{2B})^2} + \mathrm{T}^{2B} n_{\mathrm{R}}. \tag{23}$$

Using the equation for the number of particles in Eqs. (14), (18) and (19), we have that Eq. (23) can be rewritten as

$$\mu = n\mathrm{T}^{2B}\left(1 + \frac{32}{3}\sqrt{\frac{na^3}{\pi}}\right) + \delta\mu_{\mathrm{R}}, \tag{24}$$

with

$$\delta\mu_{\mathrm{R}} = -\frac{1}{V}\sum_{\mathbf{k}}{}' R_0 \frac{\epsilon_{\mathbf{k}}}{(\epsilon_{\mathbf{k}} + 2n_0 \mathrm{T}^{2B})^2} \tag{25}$$

in agreement with the zero-temperature result of Ref. [4]. Note that the beyond mean-field correction given by $\delta\mu_{\mathrm{R}}$ still contains an ultraviolet divergency, which is not related to the interaction and was thus not yet removed by the renormalization of the latter. Rather its origin lies in the fact that we have considered a $\delta-$correlated random potential. In second order perturbation theory, it changes the original chemical potential $\mu = \mu_{\mathrm{bare}}$ without the random potential, to the new value $\mu = \mu_{\mathrm{bare}} - \frac{1}{V^2}\sum_{\mathbf{k}}{}' \frac{|U_{\mathbf{k}}|^2}{\epsilon_{\mathbf{k}}}$. Thus, after averaging, we must make the change $\mu = \mu_{\mathrm{bare}} \to \mu + R_0(1/V)\sum_{\mathbf{k}}{}' 1/\epsilon_{\mathbf{k}}$, which removes the divergency. After renormalizing in this way, we have

$$\delta\mu_{\mathrm{R}} = 6\ \mathrm{T}^{2B} n_{\mathrm{R}}, \tag{26}$$

which corresponds to a shift for the macroscopic compressibility $\partial(\delta\mu_{\mathrm{R}})/\partial n = \delta\chi/\chi_0 = 3n_{\mathrm{R}}/n$ with respect to the mean-field value $\chi_0 = \mathrm{T}^{2B} n/m$ of the Bogoliubov theory for a clean system. Using the thermodynamic relation $\mu = \partial F/\partial N$, we can calculate the free energy F from Eq. (24). At $T=0$ this coincides with the energy E and we find [1, 4]

$$\frac{\langle E\rangle}{V} \simeq \frac{2\pi a\hbar^2 n^2}{m}\left[1 + \frac{128}{15}\left(\frac{na^3}{\pi}\right)^{\frac{1}{2}} + 8R_0'\right]. \tag{27}$$

III. SUPERFLUID COMPONENT

In a Bose-Einstein condensate, random impurities constitute a source of incoherent scattering which tends to localize the condensate. As shown by Huang and Meng [1] the formation of local condensates in the minima of the random potential reduces the superfluid component of the fluid even at zero temperature, where, in the absence of disorder, the whole fluid would be superfluid [13]. For the sake of completeness and in order to propose a simpler derivation, we rederive in this section the superfluid component in the presence of weak disorder [1, 2, 4] by extending the Bogoliubov diagonalization method to a moving system. The superfluid density n_s is related to the total density by the relation

$$n_s = n - n_n, \tag{28}$$

where n_n is the density of the normal component of the fluid. In order to calculate n_n , we consider the action of Eq. (7) when the gas is in motion. The moving reference system is related to the laboratory system by a Galilean transformation $\mathbf{x}' = \mathbf{x} + \mathbf{u}t$ and $t' = t$. The fields in the moving system have a relative velocity $\mathbf{v}_s$ with respect to those in the inertial frame. They are related by the transformation $\Psi^{*\prime}(\mathbf{x}', t') = e^{-im\mathbf{v}_s\mathbf{x}/\hbar}\Psi^*(\mathbf{x}, t)$ and $\Psi'(\mathbf{x}', t') = e^{im\mathbf{v}_s\mathbf{x}/\hbar}\Psi(\mathbf{x}, t)$. Therefore, in the new reference system, the action in Eq. (7) becomes

$$S[a^*, a] = \sum_{\mathbf{k},n}\left[-i\hbar\omega_n + \hbar\mathbf{k}(\mathbf{u} - \mathbf{v}_s) + \epsilon_{\mathbf{k}} - \mu_{\mathrm{eff}}\right] a^*_{\mathbf{k},n} a_{\mathbf{k},n} + \frac{1}{V}\sum_{\mathbf{k},\mathbf{k}',\mathbf{n}} a^*_{\mathbf{k},n} U_{\mathbf{k}-\mathbf{k}'} a_{\mathbf{k}',n}\delta_{n,0} + \frac{1}{2}\frac{1}{\hbar\beta V}\sum_{\substack{\mathbf{k},\mathbf{k}',\mathbf{q}\\ n,n',m}} V_{\mathbf{q}}\ a^*_{\mathbf{k}+\mathbf{q},n+m} a^*_{\mathbf{k}'-\mathbf{q},n'-m} a_{\mathbf{k}',n'} a_{\mathbf{k},n}, \tag{29}$$

where the new chemical potential is defined as $\mu_{\mathrm{eff}} = \mu + m\mathbf{u}\mathbf{v}_s - m\mathbf{v}_s^2/2$. In the broken symmetry regime, the action in the new reference frame, up to the quadratic order in the fluctuations fields, reads

$$S^{(2)}[a, a^*] = -\hbar\beta\mu_{\mathrm{eff}} n_0 V + \frac{1}{2}\hbar\beta n_0^2 V_0 V + \hbar\beta n_0 U_{\mathbf{0}} + \sum_{\mathbf{k},n}{}' \left[-i\hbar\omega_n + \hbar\mathbf{k}(\mathbf{u} - \mathbf{v}_s) + \epsilon_{\mathbf{k}} - \mu_{\mathrm{eff}} + 2n_0 V_0\right] a^*_{\mathbf{k},n} a_{\mathbf{k},n} + \left(\frac{n_0}{V}\right)^{\frac{1}{2}} \sum_{\mathbf{k},n}{}' \left(a^*_{\mathbf{k},n} U_{\mathbf{k}} + U_{-\mathbf{k}} a_{\mathbf{k},n}\right)\delta_{n,0} + \frac{1}{2} n_0 V_0 \sum_{\mathbf{k},n}{}' (a^*_{\mathbf{k},n} a^*_{-\mathbf{k},-n} + a_{\mathbf{k},n} a_{-\mathbf{k},-n}). \tag{30}$$

The latter action can again be diagonalized by the Bogoliubov transformation described in Eqs. (9)–(11). This is achieved by replacing $\mu \to \mu_{\rm eff}$ in the definition of $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ and $\hbar\Omega_{\mathbf{k}}$, and by defining the new shift variable $z_{\pm\mathbf{k}}$ as

$$z_{\pm\mathbf{k}} = (n_0/V)^{\frac{1}{2}} \frac{U_{\mathbf{k}}}{\hbar\Omega_{\mathbf{k}} \pm \hbar\mathbf{k}\,(\mathbf{u}-\mathbf{v}_s)} (u_{\mathbf{k}} - v_{\mathbf{k}})^2 . \quad (31)$$

Performing the diagonalization, the functional integration and the average over the disorder, we obtain the averaged thermodynamic potential

$$\begin{aligned}\langle\Omega\rangle = & -\mu_{\rm eff} n_0 V + \frac{1}{2} n_0^2 V_0 V \\ & + \sum_{\mathbf{k}}{}' \frac{1}{\beta} \ln\left\{1 - e^{-\beta[\hbar\Omega_{\mathbf{k}} + \hbar\mathbf{k}(\mathbf{u}-\mathbf{v}_s)]}\right\} \\ & + \frac{1}{2}\sum_{\mathbf{k}}{}' \left[\hbar\Omega_{\mathbf{k}} - (\epsilon_{\mathbf{k}} - \mu_{\rm eff} + 2n_0V_0)\right] \\ & - \sum_{\mathbf{k}}{}' n_0 R_0 \frac{(\epsilon_{\mathbf{k}} - \mu_{\rm eff} + n_0 V_0)}{\hbar\Omega_{\mathbf{k}}^2 - \left[\hbar\mathbf{k}\,(\mathbf{u}-\mathbf{v}_s)\right]^2}. \end{aligned} \quad (32)$$

Expanding for small $\hbar\mathbf{k}\,(\mathbf{u}-\mathbf{v}_s)$ to second order, we have

$$\begin{aligned}\langle\Omega\rangle \simeq & -\mu_{\rm eff} n_0 V + \frac{1}{2} n_0^2 V_0 V + \sum_{\mathbf{k}}{}' \frac{1}{\beta}\ln\left(1 - e^{-\beta\hbar\Omega_{\mathbf{k}}}\right) \\ & + \frac{1}{2}\sum_{\mathbf{k}}{}' \left[\hbar\Omega_{\mathbf{k}} - (\epsilon_{\mathbf{k}} - \mu_{\rm eff} + 2n_0V_0)\right] \\ & - \sum_{\mathbf{k}}{}' n_0 R_0 \frac{(\epsilon_{\mathbf{k}} - \mu_{\rm eff} + n_0V_0)}{\hbar\Omega_{\mathbf{k}}^2} \\ & - \sum_{\mathbf{k}}{}' \frac{\hbar^2\mathbf{k}^2}{3}\left[\beta \frac{e^{-\beta\hbar\Omega_{\mathbf{k}}}}{2\left[e^{-\beta\hbar\Omega_{\mathbf{k}}} - 1\right]} + R_0 n_0 \frac{2\epsilon_{\mathbf{k}}}{\hbar\Omega_{\mathbf{k}}^4}\right](\mathbf{u}-\mathbf{v}_s)^2 , \end{aligned} \quad (33)$$

where the term linear in $\mathbf{u}-\mathbf{v}_s$ vanishes as a consequence of the symmetry $\hbar\Omega_{\mathbf{k}} = \hbar\Omega_{-\mathbf{k}}$ of the Bogoliubov spectrum of Eq. (11). Moreover, minimizing this thermodynamic potential with respect to n_0, we obtain the zero-loop result $\mu_{\rm eff} = n_0 V_0$ which gives $\hbar\Omega_{\mathbf{k}} \simeq \sqrt{\epsilon_{\mathbf{k}}\,(\epsilon_{\mathbf{k}} + 2\mu_{\rm eff})}$. At this minimum, the momentum of the system can be calculated from the thermodynamic relation

$$\mathbf{p} = -\frac{\partial\langle\Omega\,(T,V,\mu_{\rm eff}\,(\mathbf{u})\,,\mathbf{u})\rangle}{\partial\mathbf{u}}\Big|_{T,V,\mu}. \quad (34)$$

We find

$$\begin{aligned}\mathbf{p} = & mVn\mathbf{v}_s + \frac{1}{3}\sum_{\mathbf{k}}{}' \left[\beta\hbar^2\mathbf{k}^2 \frac{e^{-\beta\hbar\Omega_{\mathbf{k}}}}{\left[e^{-\beta\hbar\Omega_{\mathbf{k}}} - 1\right]}\right. \\ & \left. + R_0\; n_0\; \hbar^2\mathbf{k}^2 \frac{4\epsilon_{\mathbf{k}}}{\hbar\Omega_{\mathbf{k}}^4}\right](\mathbf{u}-\mathbf{v}_s)\,, \end{aligned} \quad (35)$$

where we have used the thermodynamical relation $n = -(1/V)\,\partial\langle\Omega\rangle/\partial\mu$ and the identity $\partial\mu_{\rm eff}/\partial\mathbf{u} = m\mathbf{v}_s$. Therefore, we can conclude that the density of the normal part of the fluid moving with velocity $\mathbf{u}$ is given by [1]

$$n_n = \frac{1}{V}\sum_{\mathbf{k}}{}' \frac{2}{3}\;\beta\;\epsilon_{\mathbf{k}} \frac{e^{\beta\hbar\Omega_{\mathbf{k}}}}{\left[e^{\beta\hbar\Omega_{\mathbf{k}}} - 1\right]^2} + \frac{4}{3} n_{\rm R}. \quad (36)$$

Note that in two dimensions the derivation of the normal component due to the disorder is analogous, but the factor 1/3 in the formula for the thermodynamic potential in Eq. (36) is replaced by a factor 1/2. After the integration in two-dimensions, we get

$$n_n = \frac{1}{V}\sum_{\mathbf{k}}{}' \;\beta\;\epsilon_{\mathbf{k}} \frac{e^{\beta\hbar\Omega_{\mathbf{k}}}}{\left[e^{\beta\hbar\Omega_{\mathbf{k}}} - 1\right]^2} + \frac{R_0 m^2}{8\pi^3\hbar^4 a}, \quad (37)$$

in agreement with Refs. [2, 14].

The depletion of the superfluid density due to the disorder affects also the propagation of an external disturbance through the system, because the collective motion of the superfluid component is "hampered" by the component of the condensate localized in the minima of the disorder potential. In order to see this effect, we now calculate the corrections to the velocity of sound induced by the disorder at zero temperature. Let us assume that for weak disorder the dynamics of the superfluid component of the gas can be described by the phenomenological two-fluid hydrodynamic equations [15]

$$\begin{aligned} & \frac{\partial}{\partial t} n + \nabla\left(\mathbf{v}_s n_s + \mathbf{v}_n n_n\right) = 0 \\ & m\frac{\partial}{\partial t}\mathbf{v}_s + \nabla\left(\mu + \frac{1}{2}m\mathbf{v}_s^2\right) = 0, \end{aligned} \quad (38)$$

where μ is the chemical potential given in Eq. (24). However, in that expression, we shall neglect the second-order Beliaev term due to the normal interactions in comparison to the corrections due to the disorder, and we shall focus on the latter. Then Eqs. (38) represent a Landau "two-fluid" model for the superfluid condensate and the localized non-uniform condensate. What is somewhat unusual is that, in this case, the normal component induced by the disorder is at zero temperature and does not carry any entropy of the system. Furthermore, we assume that for low frequency excitations, only the superfluid component can react to the probe, while the localized normal component remains stationary. Such a situation is familiar from the propagation of the fourth sound in ^{4}He [16] which is expressed by the condition $\mathbf{v}_n = 0$. If we restrict ourselves to the linear regime, we can write $n(t) = n + \delta n(t)$ and $\mu = \mu_0 + \delta\mu$ with $\delta\mu = (\partial\mu/\partial n)\,\delta n$. Then, Eqs. (38) give the equation of motion

$$m\frac{\partial^2}{\partial t^2}\delta n - \nabla\left[n_s\nabla\left(\frac{\partial\mu}{\partial n}\delta n\right)\right] = 0. \quad (39)$$

From Eqs. (20) and (26) we have that $(\partial\mu/\partial n) = \mathrm{T}^{2B}\,(1 + 3n_{\rm R}/n)$. Moreover, from the result for the superfluid density in Eq. (28) we have $n_s = n\,(1 - 4n_{\rm R}/3n)$.

Therefore, Eq. (39) can be put into the form

$$\frac{\partial^2}{\partial t^2}\delta n - c^2\nabla^2\delta n = 0, \tag{40}$$

which exhibits a phonon dispersion $\hbar\omega = cq$. Within this direct approach, the sound velocity is found to be $c^2 \simeq c_0^2(1+5n_{\rm R}/3n)$ in agreement with Refs. [2, 4]. Note that $c_0^2 = \mathrm{T}^{2B}n/m$ is the mean-field value of the Bogoliubov theory for the clean system. The derivation of the sound mode we gave here is rather general and sufficiently simple to be generalized in order to calculate the effects of the disorder on the frequencies of the collective modes in trapped gases [17].

IV. POPOV'S THEORY

At finite temperature the interactions of the thermal component of the gas are described by the contributions to the action beyond the quadratic order given in Eq. (8). In this section we extend the Bogoliubov approach of Section II including these fluctuations according to the scheme of the Popov theory [18] which is designed for the temperature domain $k_BT > \mu$. In that approximation the cubic and quartic contributions are taken as

$$S^{(3)}[a,a^*] \simeq \sqrt{\frac{n_0}{\hbar\beta V}}{\sum_{\mathbf{k},n}}' \tilde{n}V_0\left(a^*_{\mathbf{k},n} + a_{\mathbf{k},n}\right) \tag{41}$$

and

$$S^{(4)}[a,a^*] \simeq \frac{2}{\hbar\beta V}{\sum_{\mathbf{k},n}}' \tilde{n}V_0 a^*_{\mathbf{k},n}a_{\mathbf{k},n}, \tag{42}$$

where the temperature dependent total depletion $\tilde{n} \equiv \langle\langle \psi'^{\star}\psi'\rangle_{\rm gr}\rangle$ is still to be determined. For our present thermodynamic considerations, we neglect the cubic terms following [18] and include the quartic term. The cubic term in Eq. (41) only contributes in second and higher orders of V_0 and is taken into account by the introduction of the $\mathrm{T}^{2B}-$matrix below. The new action is still diagonalized by the same Bogoliubov transformation Eqs. (9)–(11) but with the difference that the chemical potential μ has now to be replaced everywhere by the new variable

$$\mu' = \mu - 2\tilde{n}V_0, \tag{43}$$

and that the condensate density n_0 becomes strongly temperature dependent. Performing again the functional integral and the disorder average, the thermodynamic potential of Eq. (13) reads

$$\begin{aligned}\langle\Omega\rangle = &-\mu n_0 V + \frac{1}{2}n_0^2 V_0 V + {\sum_{\mathbf{k}}}' \frac{1}{\beta}\ln\left(1-e^{-\beta\hbar\Omega_{\mathbf{k}}}\right)\\ &+\frac{1}{2}{\sum_{\mathbf{k}}}'\left[\hbar\Omega_{\mathbf{k}} - (\epsilon_{\mathbf{k}} - \mu' + 2n_0V_0)\right]\\ &-{\sum_{\mathbf{k}}}' R_0 n_0\frac{(\epsilon_{\mathbf{k}} - \mu' + n_0V_0)}{\hbar\Omega_{\mathbf{k}}^2}.\end{aligned} \tag{44}$$

Popov theory is equivalent to replacing in the contribution $n_0^2V_0/2$ to the pressure Ω/V the bare interaction V_0 by the renormalized T^{2B}-matrix and to adding the contributions $2n_0\tilde{n}\mathrm{T}^{2B} + \tilde{n}^2\mathrm{T}^{2B}$ [18]. After these steps the averaged thermodynamic potential can be rewritten as

$$\begin{aligned}\langle\Omega\rangle = &-\mu n_0 V + \frac{1}{2}n_0^2\mathrm{T}^{2B}V + 2n_0\tilde{n}\mathrm{T}^{2B} + \tilde{n}^2\mathrm{T}^{2B}\\ &+{\sum_{\mathbf{k}}}' \frac{1}{\beta}\ln\left(1-e^{-\beta\hbar\Omega_{\mathbf{k}}}\right)\\ &+\frac{1}{2}{\sum_{\mathbf{k}}}'\left[\hbar\Omega_{\mathbf{k}} - \left(\epsilon_{\mathbf{k}} - \mu' + 2n_0\mathrm{T}^{2B}\right)\right]\\ &-{\sum_{\mathbf{k}}}' R_0 n_0\frac{\left(\epsilon_{\mathbf{k}} - \mu' + n_0\mathrm{T}^{2B}\right)}{\hbar\Omega_{\mathbf{k}}^2}.\end{aligned} \tag{45}$$

The equilibrium condition at fixed temperature is found by minimizing the thermodynamic potential of Eq. (45) with respect to n_0. Using the modified Hugenholtz-Pines relation $\mu' = n_0\mathrm{T}^{2B}$, which fixes $\tilde{n}$ according to Eq. (43) as $\tilde{n} = [(\mu/\mathrm{T}^{2B}) - n_0]/2$, we obtain instead of Eq. (21)

$$\begin{aligned}\mu' = &n_0\mathrm{T}^{2B} + n_0\mathrm{T}^{2B}\left(\frac{40}{3}\sqrt{\frac{n_0a^3}{\pi}}\right) +\\ &+\mathrm{T}^{2B}\frac{1}{V}{\sum_{\mathbf{k}}}'\left[\frac{2\epsilon_{\mathbf{k}} + n_0\mathrm{T}^{2B}}{\hbar\Omega_{\mathbf{k}}}N(\hbar\Omega_{\mathbf{k}}) - N(\epsilon_{\mathbf{k}})\right]\\ &+2\zeta\left(\frac{3}{2}\right)\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\mathrm{T}^{2B}\\ &-\frac{1}{V}{\sum_{\mathbf{k}}}' R_0\frac{\epsilon_{\mathbf{k}}}{\left(\epsilon_{\mathbf{k}} + 2n_0\mathrm{T}^{2B}\right)^2} + \mathrm{T}^{2B}n_{\rm R},\end{aligned} \tag{46}$$

where $n_{\rm R}$ is given by the same expression as in Eq. (19). In the "high" temperature limit $k_BT \gg n_0\mathrm{T}^{2B}$, the main contribution to the momentum integral containing the Bose distribution comes from the region $\epsilon_{\mathbf{k}} \le n_0\mathrm{T}^{2B}$ and the Bose distribution can be approximated as $N(x) \simeq k_BT/x$. Therefore, Eq. (46) can be rewritten as

$$\begin{aligned}\mu' = &n_0\mathrm{T}^{2B}\left(1 + \frac{40}{3}\sqrt{\frac{n_0a^3}{\pi}}\right)\\ &-3\mathrm{T}^{2B}\left(n_0\mathrm{T}^{2B}\right)^{\frac{1}{2}}\frac{m^{\frac{3}{2}}k_BT}{2\pi\hbar^3} + 2\zeta\left(\frac{3}{2}\right)\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\mathrm{T}^{2B}\\ &-\frac{1}{V}{\sum_{\mathbf{k}}}' R_0\frac{\epsilon_{\mathbf{k}}}{\left(\epsilon_{\mathbf{k}} + 2n_0\mathrm{T}^{2B}\right)^2} + \mathrm{T}^{2B}n_{\rm R}.\end{aligned} \tag{47}$$

In the limit of zero disorder $R_0 = 0$ this result reduces, of course, to that of Popov [18]. The new equation of state

is

$$n = -\frac{1}{V}\frac{\partial\langle\Omega\rangle}{\partial\mu} = n_0 + \frac{8}{3}\left(\frac{n_0 a^3}{\pi}\right)^{\frac{1}{2}} + \frac{1}{V}{\sum_{\mathbf{k}}}' \frac{\epsilon_{\mathbf{k}} + n_0 \mathrm{T}^{2B}}{\hbar\Omega_{\mathbf{k}}} N(\hbar\Omega_{\mathbf{k}}) + \frac{1}{V}{\sum_{\mathbf{k}}}' \frac{R_0 n_0}{\left(\epsilon_{\mathbf{k}} + 2n_0\mathrm{T}^{2B}\right)^2}. \quad (48)$$

This latter equation has the same form as the equation of state of the Bogoliubov theory as given in Eq. (14) with (18) and (19), but the domain of validity and the details of the temperature dependence are, of course, different. In the "high" temperature region $k_B T \gg n_0 \mathrm{T}^{2B}$ where the Popov theory applies, we can neglect the quantum depletion of the zero-temperature theory, and the thermal depletion $n' = (1/V)\sum_{\mathbf{k}}' N(\hbar\Omega_{\mathbf{k}})\left[\left(\epsilon_{\mathbf{k}} + n_0\mathrm{T}^{2B}\right)/\hbar\Omega_{\mathbf{k}}\right]$ can be simplified as

$$n' \simeq -\left(n_0 \mathrm{T}^{2B}\right)^{\frac{1}{2}} \frac{m^{\frac{3}{2}} k_B T}{2\pi\hbar^3} + \zeta\left(\frac{3}{2}\right)\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{\frac{3}{2}}. \quad (49)$$

Therefore, Eq. (48) can be rewritten as

$$n = n_0 + n\left(\frac{T}{T_c^0}\right)^{\frac{3}{2}} - \left(n_0\mathrm{T}^{2B}\right)^{\frac{1}{2}} \frac{m^{\frac{3}{2}} k_B T}{2\pi\hbar^3} + n_{\mathrm{R}}. \quad (50)$$

The curve for the critical temperature T_c in the Popov theory can be obtained by putting $n_0(T)$ equal to zero in Eq. (50). We note that the contribution in Eq. (50) due to the disorder vanishes when approaching the critical point because, according to Eq. (20), we have $n_{\mathrm{R}} \propto \sqrt{n_0(T)}$. Therefore, we can argue that even the presence of a random potential of the kind under consideration here, the Popov approximation does not shift the value of the critical temperature away from that of an ideal Bose gas T_c^0. In the absence of disorder, the Popov approximation of a dilute Bose gas describes a first-order phase transition: at the critical temperature T_c^0 the condensate density $n_0(T)$ exhibits a discontinuous jump to a finite value. The latter can be calculated analytically [20] as a function of the scattering length a. It can be shown, that the effect of the disorder is to suppress this discontinuity and that the jump vanishes for R_0' larger than some value $\tilde{R}_0'$.

Nevertheless, the transition to superfluidity occurs at the temperature T_s where the superfluid density vanishes, which means, where the equation

$$n = n_n = \frac{1}{V}{\sum_{\mathbf{k}}}' \frac{2}{3}\ \beta_s\ \epsilon_{\mathbf{k}} \frac{e^{\beta_s\hbar\Omega_{\mathbf{k}}}}{\left[e^{\beta_s\hbar\Omega_{\mathbf{k}}} - 1\right]^2} + \frac{4}{3} n_{\mathrm{R}}, \quad (51)$$

is satisfied. At "high" temperatures such that $k_B T_s \gg n_0\mathrm{T}^{2B}$, the latter equation can be approximated as

$$n \simeq n\left(\frac{T_s}{T_c^0}\right)^{\frac{3}{2}} - \frac{2}{3}\left(n_0\mathrm{T}^{2B}\right)^{\frac{1}{2}} \frac{m^{\frac{3}{2}} k_B T_s}{2\pi\hbar^3} + \frac{4}{3} n_{\mathrm{R}}. \quad (52)$$

Therefore, Eqs. (50) and (52) represent two coupled equation for n_0 and T_s. In the lowest order in the strength of disorder and interaction we can approximate Eq. (50) with the ideal gas result $n_0 \simeq n\left[1 - \left(T/T_c^0\right)^{(3/2)}\right]$. Substituting this in Eq. (52) and neglecting there the second term on the right-hand side, in the limit $R_0' \gg \left(na^3\right)^{1/6}$, we obtain an equation for the critical temperature T_s as a function of the total density and the strengths of the interaction and the disorder

$$\left[1 - \left(T_s/T_c^0\right)^{\frac{3}{2}}\right]^{\frac{1}{2}} \simeq \frac{4}{3}\sqrt{\frac{1}{na}} \frac{R_0 m^2}{8\ \pi^{3/2}\ \hbar^4} \equiv \frac{4}{3} R_0'. \quad (53)$$

Solving for T_s we find for the critical temperature where superfluidity disappears when coming from lower temperatures

$$T_s/T_c^0 \simeq 1 - (32/27)\, {R_0'}^2. \quad (54)$$

This result coincides with Eq. (32) of Ref. [4]. Here it has been derived assuming that near the critical temperature T_s the interaction between the bosonic particles can be described by the temperature independent $\mathrm{T}^{2B}-$matrix. This is consistent only if the critical temperature T_s is not too close to the transition temperature T_c^0. The reason is that near the Bose-Einstein condensation point the quasi-particle interactions acquire a strong temperature dependence [7, 20, 21]. As pointed out in Ref. [4], the consistency condition is satisfied when $R_0' \gg \left(na^3\right)^{1/6}$. In the opposite regime, when $R_0' \ll \left(na^3\right)^{1/6}$, we will show in the next section that we have $T_s \simeq T_c$, where T_c is the Bose-Einstein condensation temperature including disorder effects which depends linearly on R_0. In that case, temperature effects on the particle-particle scattering cannot be neglected and the two-body $\mathrm{T}^{2B}-$matrix of the Popov theory has to be replaced by the many-body $\mathrm{T}-$matrix, which we shall do in the next section.

The failure of the Popov approximation in the regime $R_0' \ll \left(na^3\right)^{1/6}$ has a specific physical reason. The Popov approximation, as considered here up to now, neglects the Hartree-Fock contribution by which the presence of the impurities affects the scattering among two thermal particles. This can be seen, for example, by calculating the relation between the chemical potential μ and the total density n and comparing it with the result of Lopatin and Vinokur in Ref. [4]. Using Eqs. (47)–(49), we find

$$\mu = \mathrm{T}^{2B} n - \mathrm{T}^{2B}\frac{1}{\pi}\left(\mu'\right)^{\frac{1}{2}} \frac{m^{\frac{3}{2}} k_B T}{\hbar^3} + \mathrm{T}^{2B}\zeta\left(\frac{3}{2}\right)\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{\frac{3}{2}} - \frac{1}{V}{\sum_{\mathbf{k}}}' R_0 \frac{\epsilon_{\mathbf{k}}}{\left(\epsilon_{\mathbf{k}} + 2\mu'\right)^2}. \quad (55)$$

We observe that in comparison with the theory of Ref. [4] we seem to miss in Eq. (55) the Hartree-Fock term $\mathrm{T}^{2B} R_0 m^3 k_B T / 4\pi^2\hbar^6$. A corresponding contribution seems also to be absent in the equation of state in Eq. (50)

and in the normal density component of Eq. (52). However, since we are in the regime $R_0' \gg \left(na^3\right)^{1/6}$, the latter contribution is in fact negligible and one obtains the value of T_s as indicated in Eq. (54). In order to access also the regime $R_0' \leq \left(na^3\right)^{1/6}$, in the next section, we will extend the Popov theory to the many-body T−matrix approximation. In this way we will find results similar to the one of Lopatin and Vinokur but within a gapless theory approaching the critical point from below.

V. MANY-BODY T-MATRIX

In the vicinity of the critical temperature of Bose-Einstein condensation, the interactions between the quasi-particles are strongly renormalized by temperature effects and vanish at the transition point for this reason [21]. For a clean system, Bijlsma and Stoof [7] have shown that the quasi-particles interaction is well described by the many-body T-matrix T^{MB}. In the presence of weak disorder, the many-body T-matrix continues to obey, in lowest order, the same formal Bethe-Salpeter equation as in the absence of disorder [7]

$$\begin{aligned}\mathrm{T}^{MB}(\mathbf{k},\mathbf{k}',\mathbf{K};z) &= V(\mathbf{k}-\mathbf{k}') + \int \frac{d\mathbf{k}''}{(2\pi)^3} V(\mathbf{k}-\mathbf{k}'') \\ &\times \left\{ \left[\frac{u_+^2 u_-^2}{z-\hbar\Omega_+ - \hbar\Omega_-} - \frac{v_+^2 v_-^2}{z+\hbar\Omega_+ + \hbar\Omega_-} \right] (1+N_+ + N_-) \right. \\ &+ \left. \left[\frac{u_-^2 v_+^2}{z+\hbar\Omega_+ - \hbar\Omega_-} - \frac{u_+^2 v_-^2}{z-\hbar\Omega_+ + \hbar\Omega_-} \right] (N_+ - N_-) \right\} \\ &\times \mathrm{T}^{MB}(\mathbf{k}'',\mathbf{k}',\mathbf{K};z), \end{aligned} \tag{56}$$

where $N_+ \equiv N(\hbar\Omega_+)$ and $N_- \equiv N(\hbar\Omega_-)$. The plus sign denotes the momentum argument $\mathbf{K}/2+\mathbf{k}''$, and similarly the minus sign denotes the argument $\mathbf{K}/2-\mathbf{k}''$. In the low-temperature domain, where BEC-experiments with cold atoms are always realized, the momentum and energy dependence of the T^{MB}-matrix can be neglected. This even applies to the domain $k_B T_c \gg k_B T \gg \mu$ of sufficiently high temperatures where the Popov approximation applies, to which we sometimes refer as the "high"-temperature domain in this context. Then we have

$$\begin{aligned}\mathrm{T}^{MB}(\mathbf{0},\mathbf{0},\mathbf{0};0) &= V_0 - V_0 T^{MB}(\mathbf{0},\mathbf{0},\mathbf{0};0) \qquad (57)\\ &\times \int \frac{d\mathbf{k}}{(2\pi)^3} \left[\frac{1}{2\hbar\Omega_{\mathbf{k}}} + \frac{n_0 V_0}{4\left(\hbar\Omega_{\mathbf{k}}\right)^3} \right] \left[1+2N(\hbar\Omega_{\mathbf{k}})\right].\end{aligned}$$

The integral on the right hand side contains an infrared divergency [22] caused by the well-known fact that the finite-temperature theory does not properly account for the dynamics of the phase fluctuations in the infrared limit [19]. However, in the "high" temperature limit $na\lambda_{\mathrm{th}}^2 \lesssim 1$ the infrared divergent term must, in fact, be dropped. This follows because at "high" temperatures the Bogoliubov spectrum $\hbar\Omega_{\mathbf{k}}$ deviates from $\epsilon_{\mathbf{k}} - \mu'$ only for a very small interval of momenta around zero, where $\epsilon_{\mathbf{k}} \lesssim n_0 \mathrm{T}^{2B} < n\mathrm{T}^{2B}$. Therefore this phononic part of the spectrum merely makes an asymptotically small contribution to all thermodynamic quantities. Therefore, we can set $u_{\mathbf{k}} = 1$ and $v_{\mathbf{k}} = 0$ in Eq. (56). With this, and after having eliminated the bare potential V_0 by means of the Lippmann-Schwinger equation for the T^{2B}−matrix, we obtain

$$\begin{aligned}\mathrm{T}^{MB}(\mathbf{0},\mathbf{0},\mathbf{0};0) &= \mathrm{T}^{2B}(\mathbf{0},\mathbf{0};-2\mu') - \mathrm{T}^{2B}(\mathbf{0},\mathbf{0};-2\mu') \\ &\times \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{N(\hbar\Omega_{\mathbf{k}})}{\hbar\Omega_{\mathbf{k}}} \mathrm{T}^{MB}(\mathbf{0},\mathbf{0},\mathbf{0};0).\end{aligned} \tag{58}$$

Solving for $\mathrm{T}^{MB}(\mathbf{0},\mathbf{0},\mathbf{0};0)$ we find [7, 20]

$$\mathrm{T}^{MB}(\mathbf{0},\mathbf{0},\mathbf{0};0) = \frac{\mathrm{T}^{2B}(\mathbf{0},\mathbf{0};-2\mu')}{1+\mathrm{T}^{2B}(\mathbf{0},\mathbf{0};-2\mu') \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{N(\hbar\Omega_{\mathbf{k}})}{\hbar\Omega_{\mathbf{k}}}}. \tag{59}$$

The many-body T-matrix approximation in a dilute ultracold Bose gas can now be obtained from the formulas of the Popov theory as developed in the previous section just by replacing [20] there the two-body T-matrix $\mathrm{T}^{2B}(\mathbf{0},\mathbf{0};-2\mu') \equiv \mathrm{T}^{2B}$ by the temperature dependent many-body T-matrix $\mathrm{T}^{MB}(\mathbf{0},\mathbf{0},\mathbf{0};0)$. The excitation spectrum is thereby changed to

$$\hbar\Omega_{\mathbf{k}} = \sqrt{\left[\epsilon_{\mathbf{k}} - \mu + \hbar\Sigma_{11}(\mathbf{k},\omega_n)\right]^2 - \left[\hbar\Sigma_{12}(\mathbf{k},\omega_n)\right]^2}, \tag{60}$$

where the self-energies are given by

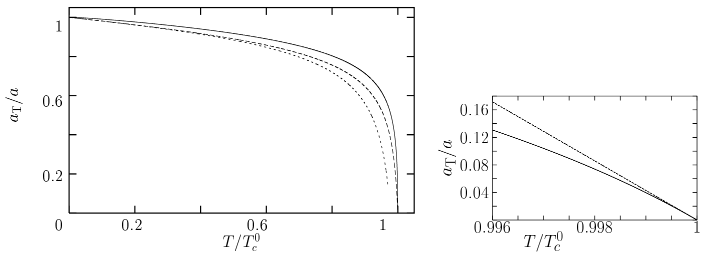
$$\begin{aligned}\hbar\Sigma_{12} &= n_0 \mathrm{T}^{MB}(\mathbf{0},\mathbf{0},\mathbf{0};0)\,, \\ \hbar\Sigma_{11} &= 2n T^{MB}(\mathbf{0},\mathbf{0},\mathbf{0};0).\end{aligned} \tag{61}$$

The condition $\mu' = \mu - \hbar\Sigma_{11} = n_0 \mathrm{T}^{MB}(\mathbf{0},\mathbf{0},\mathbf{0};0)$ ensures the spectrum to be gapless.

The Bose factor in the many-body T−matrix of Eq. (59) leads to a temperature-dependent scattering length, defined as

$$a_{\mathrm{T}} \equiv m\mathrm{T}^{MB}\left(\mathbf{0},\mathbf{0},\mathbf{0};0\right)/4\pi\hbar^2. \tag{62}$$

For temperatures not too close to the Bose-Einstein critical temperature, where the usual mean-field Popov theory of the previous section is valid, the denominator in Eq. (59) represents a negligible correction. In that case, the many-body T−matrix indeed reduces to the two-body temperature-independent T^{2B}−matrix $\mathrm{T}^{2B} = 4\pi\hbar^2 a/m$. However, approaching the critical region near the Bose-Einstein transition, the scattering between quasi-particles now becomes strongly temperature dependent [7, 20]. A more detailed insight can be gained by considering the "high" temperature expansion of Eq. (59) [20]. In that case, the latter reduces to a quadratic equation which can be solved analytically. Using Eq. (62) we find for the temperature dependent scat-



(a) (b)

Figure 1: (a) Temperature dependent scattering length as a function of the rescaled temperature T/T_c^0 calculated numerically from Eq. (59) (solid line) and from the analytic expression of Eq. (63) (dashed line). The dotted line shows the asymptotic limit described by Eq. (64) valid in the Popov region for temperatures not too close to T_c^0. (b) Temperature dependent scattering length as a function of the rescaled temperature T/T_c^0 near T_c^0. The lower line shows the numerical result from Eq (59) compared to the asymptotic expression given in Eq. (65) described by the upper line. In both pictures the gas parameter of the Bogoliubov theory has been chosen such as $(na^3)^{1/3} \sim 0.01$.

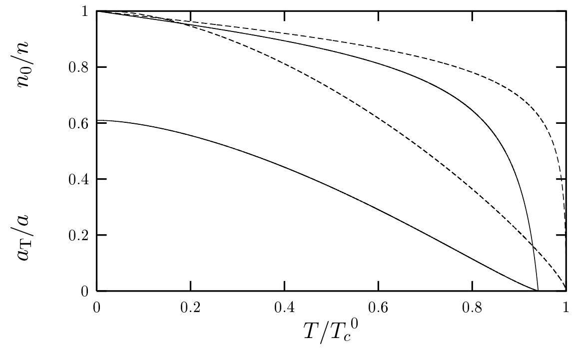


Figure 2: The lower and the upper dashed lines describe the condensate density n_0 and the scattering length $a_{\rm T}$ obtained by solving self-consistently Eqs. (63) and (66) in absence of disorder, *i.e.*, when $R = 0$. In this case, both quantities vanish at $T = T_c^0$ and no shift of the critical temperature is induced [7, 20]. The gas parameter of the Bogoliubov theory has been chosen the same as in Figure 1. The solid lines show the solution for finite disorder. We see that the temperature at which they vanish is shifted with respect to the critical temperature of the ideal gas. In order to make clearly visible the effect we have considered in the figure the case $R_0' = 0.5$. However, for such a value the condensate depletion due to the disorder at zero temperature is already $\sim 0.4\%$ of the total density. Note also that, in contrast with Popov theory, for each value of R_0' the curve for n_0 obtained from the many-body T$-$matrix approximation exhibits a second-order phase transition.

tering length

$$a_{\rm T} \simeq a \left[1 + \frac{\left(\varrho \frac{T}{T_c^0}\right)^2 - \left(\varrho \frac{T}{T_c^0}\right) \sqrt{\left(\varrho \frac{T}{T_c^0}\right)^2 + 4\frac{n_0}{n}}}{2\frac{n_0}{n}} \right]. \quad (63)$$

where $\varrho \equiv \left[2\sqrt{\pi}/\zeta\left(3/2\right)^{\frac{2}{3}}\right]\left(na^3\right)^{1/6}$. In Figure 1 the curve resulting from the"high" temperature expansion of Eq. (63) is compared with the curve obtained from Eqs. (59) and (62). For temperatures not too close to the critical point, we have $\left(\varrho T/T_c^0\right)^2 \ll 4(n_0/n)$ and the curve of Eq. (63) can be approximated by

$$a_{\rm T} \simeq a \left[1 - 2\frac{\sqrt{\pi}}{\zeta\left(3/2\right)^{\frac{2}{3}}} \left(na^3\right)^{1/6} \frac{T}{T_c^0} \frac{1}{\sqrt{n_0/n}} \right], \quad (64)$$

while for temperatures just below the critical temperature T_c^0 the scattering length between quasi-particles becomes a universal function of the density and temperature given by

$$a_{\rm T} \simeq \frac{\zeta\left(3/2\right)^{\frac{4}{3}}}{4\pi} \frac{n_0}{n^{4/3}}, \quad (65)$$

which vanishes at the critical temperature T_c^0 [7, 21] with the same power as the condensate density. The two different asymptotic regimes of the expression in Eq. (63) are also shown in Figure 1. Note that Figure 1(a) includes also the zero-temperature limit where the Popov approximation as well as the many-body T$-$matrix are infrared divergent. However, this is an asymptotic small region because the condition of validity of the Popov approximation defined by $na\lambda_{\rm th}^2 \lesssim 1$ is equivalent to impose the condition $T/T_c^0 \gtrsim [\zeta\left(3/2\right)^{\frac{2}{3}}/4\pi](na^3)^{1/3}$.

By applying these results to the equation of state in Eq. (50) the latter can be rewritten as

$$n = n_0 + n\left(\frac{T}{T_c^0}\right)^{\frac{3}{2}} - \left[n_0 \frac{4\pi\hbar^2 a_{\rm T}}{m}\right]^{\frac{1}{2}} \frac{m^{\frac{3}{2}} k_B T}{2\pi\hbar^3} + R_0 \sqrt{\frac{n_0}{a_{\rm T}}} \frac{m^2}{8\ \pi^{3/2}\ \hbar^4}. \quad (66)$$

For a clean system neither the two-body nor the many-body T-matrix theory induce a shift of the critical temperature [7]. In contrast to this, we find that in the presence of delta correlated disorder the effects described by the many-body T-matrix induce such a shift of T_c. This is illustrated in Figure 2, where the solutions of the coupled equations (63) and (66) for $a_{\rm T}$ and the condensate density n_0 are shown as function of the temperature in absence and in presence of disorder. The origin of the shift becomes evident when considering the limit $T \to T_c^0$ in Eq. (66). If we insert in that equation for the temperature dependence of the condensate density that of an ideal Bose gas, the term due to the disorder does not vanish at T_c^0 as it would in the simple Popov theory. This follows because the temperature dependent scattering length $a_{\rm T}$ goes to zero with the same exponent as the condensate density n_0, and both effects cancel. Thus we get

$$n = n\left(T/T_c^0\right)^{\frac{3}{2}} + R_0 m^3 k_B T_c^0/8\pi^2\hbar^6, \quad (67)$$

which is solved by

$$T_c \simeq T_c^0 \left(1 - \eta\right), \tag{68}$$

where $\eta \equiv R_0 m^3 k_B T_c^0 / 12\pi^2 \hbar^6 n = 2\pi/[3\zeta\left(3/2\right)^{2/3} dn^{1/3}] \ll 1$. This value differs from the Hartree-Fock result found by Lopatin and Vinokur in Ref. [4] by a factor of 1/2. Note, however, that in contrast to Ref. [4], our gapless T^{MB}-matrix theory approaches the critical point from below. In fact, the theoretical descriptions on the two sides of the critical points are remarkably different as a consequence of the difference between the two phases on both sides. The result of Lopatin and Vinokur has been recently confirmed by Zobay in Ref. [6] by means of a one-loop Wilson renormalization group calculation. This latter method approaches the critical point from above as well but takes into account critical fluctuations which are non-perturbative in the interaction. The numerical solution of the renormalization group equations shows that the critical fluctuations lead only to small corrections with respect to the result of Ref. [4].

The many-body T−matrix approximation is also useful to understand the dependence on the disorder of the superfluid transition T_s. Equation (52) can be rewritten as

$$n \simeq n \left(\frac{T_s}{T_c^0}\right)^{\frac{3}{2}} - \frac{2}{3}\left[n_0 \mathrm{T}^{MB}\right]^{\frac{1}{2}} \frac{m^{\frac{3}{2}} k_B T_s}{2\pi\hbar^3} + \frac{4}{3} R_0 \sqrt{\frac{n_0}{a_{\mathrm{T}}}} \frac{m^2}{8\ \pi^{3/2}\ \hbar^4}. \tag{69}$$

Eq. (69) together with Eqs. (63) and (66) evaluated at $T = T_s$ constitute a set of closed equations for T_s, $n_{0|T=T_s}$ and $a_{\mathrm{T}|T=T_s}$. The self-consistent solution of the three coupled equations is shown in Figure 3. When $R_0' \gg \left(na^3\right)^{1/6}$ we have that T_s lies well below T_c^0 in the domain, where the many-body T−matrix is given essentially by the temperature independent T^{2B}-matrix. In that limit T_s is given by Eq. (54) of the Popov theory. However, when $R_0' \leq \left(na^3\right)^{1/6}$, one expects $T_s \lesssim T_c^0$ and the temperature effects on the quasi-particles scattering become important. In that case, we can use the expansion of the many-body T−matrix near the critical point given in Eq. (65). From Eq. (69) to the lowest order in the disorder and in the normal interaction this leads to $T_s \simeq T_c^0\left(1 - 4\eta/3\right)$, which is smaller than the critical temperature T_c given in Eq. (68) due to the factor 4/3.

Therefore, we find that the presence of disorder shifts the critical temperature for the superfluid transition below the line of the Bose-Einstein condensation temperature. The shift increases monotonically as a function of the disorder strength R_0. For $R_0' \leq \left(na^3\right)^{1/6}$ the shift is linear in R_0. At larger values of the disorder strength, where $R_0' \gg \left(na^3\right)^{1/6}$, it becomes quadratic in R_0. The crossover between the two regimes is described accurately in Figure 3. Note that, in that figure, the results obtained in the region $R_0' \sim 1$ have to be considered as an extrapolation to the more complicate regime of strong disorder [23, 24].

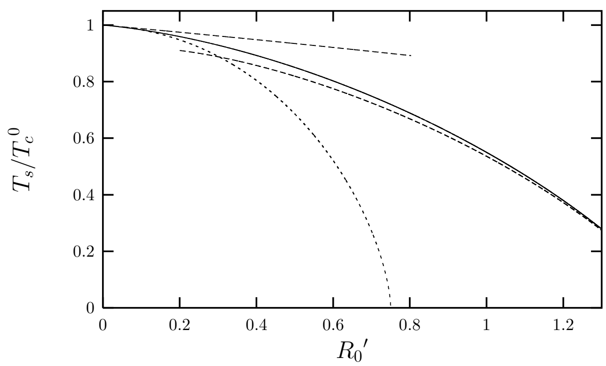


Figure 3: The solid line indicates the critical temperature T_s of the normal to superfluid transition as a function of the dimensionless disorder parameter R_0' obtained by solving self-consistently Eq. (63), Eq. (66), and Eq. (69) for $(na^3)^{1/3} = 0.01$. The upper dashed line is obtained by approximating Eq. (63) with the asymptotic solution of Eq. (65) which is valid when $T_s \simeq T_c$ for $R_0' \leq \left(na^3\right)^{1/6}$. In contrast, the lower dashed line describes the regime $R_0' \geq \left(na^3\right)^{1/6}$ where $T_s \ll T_c$ and Eq. (63) can be approximated with Eq. (64). The comparison between these three curves describes clearly how the temperature effects in the many-body T^{MB}−matrix determine the crossover from the quadratic regime to the linear regime for the dependence of T_s/T_c^0 on the disorder coupling constant R_0'. In addition, the short-dashed line shows the effects of neglecting the self-consistency in the equation of state (66), where the condensate density is approximated with the ideal gas result $n_0 = n[1-(T_s/T_c^0)^{3/2}]$. For $R_0' \geq \left(na^3\right)^{1/6}$ the short-dashed line coincides with the analytical result of the Popov theory given by Eq. (54).

Moreover, the superfluid density can be calculated from the relation $n_s = n - n_n$ with the normal density given by the right-hand side of Eq. (69) evaluated at temperatures $T < T_s$. In contrast with the Bogoliubov approach of Ref. [1], our theory includes important finite-temperature correlations between quasi-particles and we find that the superfluid density decreases monotonically as function of the temperature. This result is in agreement with the diagrammatic theory based on the replica method developed in Ref. [4].

Here, we have not discussed the damping of the sound due to the impurity scattering. This damping has so far been studied only at zero temperature in Refs. [2, 4]. The many-body T−matrix theory developed here, neglects the finite lifetime of the quasi-particles and cannot describe damping phenomena for this reason.

VI. CONCLUSIONS AND OUTLOOK

In conclusion, we have extended in this paper the perturbative approach developed by Huang and Meng in Ref. [1] for a homogeneous superfluid dilute Bose gas in the presence of weak disorder. We have shown that such an extension to finite temperatures is achieved via a suitable combination of the mean-field Popov approximation and the many-body T−matrix approximations. In particular this allows to make contact with results of second-order perturbation theory at finite temperature developed in Ref. [4] by means of the replica method. E.g. the shifts of the two different critical temperatures for the appearance of a condensate density and of a superfluid density have here been calculated when coming from the low-temperature side.

The theory could have other interesting applications. Using the notion of the quasi-condensate [18], Andersen *et al.* [19] have shown that the many-body T−matrix theory can describe the thermodynamics of a clean two-dimensional Bose gas near the Kosterlitz-Thouless superfluid transition, and it seems plausible that this continues to be true also for weak disorder. Therefore, the approach presented here could be useful in order to study the long-standing problem concerning the disorder-induced shift on the superfluid transition temperature in a two-dimensional Bose gas.

Acknowledgments

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